

# Robust Eigenvector of a Stochastic Matrix with Application to PageRank

Anatoli Juditsky and Boris Polyak<sup>\*†</sup>

## Abstract

We discuss a definition of robust dominant eigenvector of a family of stochastic matrices. Our focus is on application to ranking problems, where the proposed approach can be seen as a robust alternative to the standard PageRank technique. The robust eigenvector computation is reduced to a convex optimization problem. We also propose a simple algorithm for robust eigenvector approximation which can be viewed as a regularized power method with a special stopping rule.

## 1 Introduction

There are numerous problems which can be reformulated as finding a dominant eigenvector of a stochastic matrix — computing invariant probabilities in Markov chains, various problems in econometrics, sociometry, bibliometrics, sport etc., see an excellent survey on history of such applications [3]. Recently, ranking in the Web (PageRank) attracted a lot of attention [4, 5].

Let us briefly remind this approach and discuss some problems which arise in relation to it. There are  $n$  pages (nodes of a graph) connected with outgoing and incoming links. Then  $x_i$  — score of  $i$ -th page of the Web,  $i = 1, \dots, n$  is defined as follows:  $x_i = \sum_{j \in L_i} x_j / n_j$ , where  $n_j$  is the number of outgoing links of page  $j$  and  $L_i$  is the set of pages linked to page  $i$ . Thus the score vector  $x$  satisfies the equation

$$Px = x \quad (1)$$

with  $P_{ij} = 1/n_j$  if page  $j$  cites page  $i$  and  $P_{ij} = 0$  otherwise. I.e.  $x$  is an eigenvector corresponding to eigenvalue 1 of the (column) stochastic matrix  $P$  of size  $n \times n$ .

In the example below  $n = 7$ , outgoing links are denoted by arrows. The corresponding matrix  $P$  and

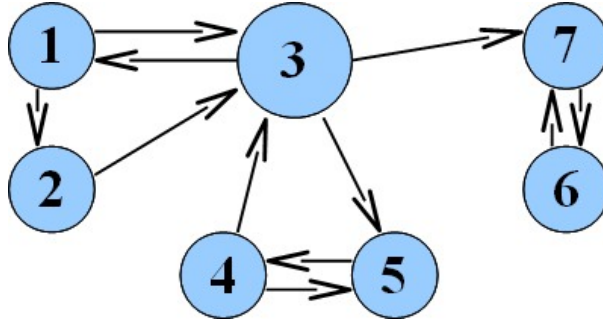


Fig. 1: Test example

its dominant (or principal) eigenvector  $\bar{x}$  are

<sup>\*</sup>A. Juditsky is with LJK, Université J. Fourier, BP 51, 38041 Grenoble Cedex 09, France [juditsky@imag.fr](mailto:juditsky@imag.fr)

<sup>†</sup>B. Polyak is with Institute for Control Sciences, 65 Profsoyuznaya str., 117997 Moscow, Russia [boris@ipu.ru](mailto:boris@ipu.ru)

$$P = \begin{pmatrix} 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\bar{x} = (0, 0, 0, 0, 0, 0.5, 0.5)^T.$$

We see that the scores of pages 6 and 7 equal 0.5 while all other pages have zero scores. Of course it contradicts common sense and our intuition about “true scores”. There are several problems related to the above definition:

1. We have assumed that  $n_j \neq 0$  for all  $j$ . Yet, in applications, there are nodes with no outgoing links — *dangling nodes* (e.g. pdf files). One can easily tackle this problem by taking  $n_j = n$  for all dangling nodes. However more complicated problem of “traps” (absorbing nodes 6 and 7 in the above example) remains open.
2. In the case of disconnected subgraphs (typical for real-life problems)  $\bar{x}$  is not unique.
3. Small changes in  $P$  may result in dramatic perturbations of  $\bar{x}$  — the computed scores are non-robust.
4. Power method with iterations  $x_{k+1} = Px_k$  do not converge for cyclic matrices; for acyclic ones it may converge slowly due to the small gap  $1 - |\lambda_2|$  (here  $\lambda_2$  is the second eigenvalue of  $P$ ).

PageRank medicine for these problems goes back to the pioneering paper [4]: allow the  $i$ -th page to receive a part of its score “for free” – replace the dominant eigenvector  $\bar{x}$  of  $P$  with the (unique) solution to the equation

$$x^\alpha = \alpha Px^\alpha + (1 - \alpha)e, \quad e_i = 1/n, \quad i = 1, \dots, n.$$

In other words, the “true” adjacency matrix  $P$  is replaced with

$$M = \alpha P + (1 - \alpha)E, \quad E_{ij} = 1/n, \quad i, j = 1, \dots, n, \quad (2)$$

so that  $x^\alpha = Mx^\alpha$  is the unique dominant eigenvector of stochastic matrix  $M$ . Unlike a dominant eigenvector of  $P$ ,  $x^\alpha$  is robust to variations in links, with fast convergence of power iterations (as  $\alpha^k$ ). As a justification the model of *surfer in the Web* is proposed in [4]: a surfer follows links with probability  $\alpha$  and jumps to an arbitrary page with probability  $1 - \alpha$ . Then  $x_i^\alpha$  is the (asymptotically) average time share he spends on page  $i$ .

Here we propose a different modification of the original score definition (1). Note that in most of applications nominal matrices are known approximately and they are subject to perturbations, so robustness issues become important. Our goal is to provide a robust counterpart to standard techniques in the spirit of Robust Optimization framework [7]. In the hindsight, it shares many common features with Robust Least Squares [6]. To the best of our knowledge this approach to ranking problems is new.

## 2 Problem formulation

We say that matrix  $P \in \mathbb{R}^{n \times n}$  is *stochastic* (column-stochastic) if  $P_{ij} \geq 0 \quad \forall i, j, \quad \sum_i P_{ij} = 1 \quad \forall j$ . The set of all stochastic matrices is denoted  $\mathcal{S}$ . The celebrated Perron-Frobenius theorem states that there exist a *dominant* eigenvector  $\bar{x} \in \Sigma$  (here  $\Sigma = \{u \in \mathbb{R}^n \mid \sum_i u_i = 1, u_i \geq 0\}$  being the standard simplex of  $\mathbb{R}^n$ ) of the “nominal” matrix  $P$ :

$$P\bar{x} = \bar{x}.$$

We are looking for a robust extension of the dominant eigenvector. Let  $\mathcal{P} \subset \mathcal{S}$  stand for the set of *perturbed* stochastic matrices  $F = P + \xi$ , where  $\xi$  is a perturbation. The function  $\max_{F \in \mathcal{P}} \|Fx - x\|$ , where  $\|\cdot\|$  is some norm, can be seen as a measure of “goodness” of a vector  $x$  as common dominant

eigenvector of the family  $\mathcal{P}$ . We say that the vector  $\hat{x}$  is a robust solution of the eigenvector problem on  $\mathcal{P}$  if

$$\hat{x} \in \underset{x \in \Sigma}{\text{Argmin}} \left\{ \max_{F \in \mathcal{P}} \|Fx - x\| \right\}. \quad (3)$$

Let us consider some choices of the parameters of the above definition – uncertainty set  $\mathcal{P}$  and the norm  $\|\cdot\|$  – for the PageRank problem. Recall that our motivation is to “immunize” the score  $\hat{x}$  against small perturbations of the adjacency matrix  $P$ . We may consider, that the uncertainty of the  $j$ -th column of  $P$ , in terms of the  $\ell_1$ -norm of elements, is bounded by  $\varepsilon_j$ . In other words, if  $[\xi]_j$  is the  $j$ -th column of the perturbation matrix, then  $\|[\xi]_j\|_1 \leq \varepsilon_j$ ,  $j = 1, \dots, n$ . From now on we assume that  $\varepsilon_j > 0$ ,  $j = 1, \dots, n$ . Further, we may fix the “total uncertainty budget”  $\varepsilon \geq \|\xi\|_1$ , where with some notational abuse,  $\|\xi\|_1 = \sum_{ij} |\xi_{ij}|$ . With a “natural choice” of the norm  $\|\cdot\| = \|\cdot\|_1$ , the definition (3) now reads

$$\begin{aligned} \hat{x}^{(1)} \in \underset{x \in \Sigma}{\text{Argmin}} \left\{ \max_{\xi} \|(P + \xi)x - x\|_1 \right. \\ \left. e^T[\xi]_j = 0, \|[\xi]_j\|_1 \leq \varepsilon_j, \|\xi\|_1 \leq \varepsilon \right\} \end{aligned} \quad (4)$$

(here  $e_i = n^{-1}$ ,  $i = 1, \dots, n$ ). Let  $n_j$  be the number of outgoing links on page  $j$ . If we set  $\varepsilon_j \leq \frac{1}{n_j}$ , then  $F = P + \xi$  is a stochastic matrix.

A different robust eigenvector may be obtained if, instead of fixing a bound for the  $\ell_1$ -norm of the perturbation, we bound its Frobenius norm:  $\|\xi\|_F \leq \varepsilon$ . When combining it with the choice of the norm  $\|\cdot\| = \|\cdot\|_2$  in (3), we come to the definition

$$\begin{aligned} \hat{x}^{(2)} \in \underset{x \in \Sigma}{\text{Argmin}} \left\{ \max_{\xi} \|(P + \xi)x - x\|_2 \right. \\ \left. e^T[\xi]_j = 0, \|[\xi]_j\|_1 \leq \varepsilon_j, \|\xi\|_F \leq \varepsilon \right\}. \end{aligned} \quad (5)$$

An interesting simple version of the problem (5) may be obtained when substituting the “column-wise constraints” on  $\xi$  with the requirement that  $P + \xi$  is stochastic:

$$\begin{aligned} \hat{x}^{(F)} \in \underset{x \in \Sigma}{\text{Argmin}} \left\{ \max_{\xi} \|(P + \xi)x - x\|_2 \right. \\ \left. F + \xi \text{ is stochastic, } \|\xi\|_F \leq \varepsilon \right\}. \end{aligned} \quad (6)$$

Obviously, there is a vast choice of uncertainty sets and norms, which will result in different definitions of robust eigenvector. Three definitions above may be viewed as a starting point for this study.

### 3 Theoretical analysis

Observe that the min-max problems involved in the definitions (4) – (6) of the robust eigenvector are not convex-concave. Indeed, if  $\Xi$  is the corresponding set of perturbation matrices, the function

$$\phi_{\Xi}(x) = \max_{\xi \in \Xi} \|(P + \xi)x - x\|$$

cannot be computed efficiently. However, one can easily upper bound the optimal values of these problems by those of simple convex problems as follows.

**Proposition 1:** Let

$$\Xi_1 = \{\xi \in \mathbb{R}^{n \times n} \mid e^T[\xi]_j = 0, \|[\xi]_j\|_1 \leq \varepsilon_j, \|\xi\|_1 \leq \varepsilon\},$$

and let  $\|\cdot\| = \|\cdot\|_1$ . Then

$$\phi_{\Xi_1}(x) \leq \varphi_1(x), \quad \varphi_1(x) = \|Px - x\|_1 + \varepsilon g_1(x),$$

where the convex function  $g_1$  is defined as

$$g_1(x) = \min_{x=u+v} \left\{ \|u\|_{\infty} + \sum_j \frac{\varepsilon_j}{\varepsilon} |v|_j \right\}.$$

The proofs of the statements of this section are postponed till the appendix.

Note that  $g_1$  is a norm. Further, when  $\varepsilon_1 = \dots = \varepsilon_n$  and  $s = \frac{\varepsilon}{\varepsilon_1}$  is integer,  $g_1(x)$  is simply the sum of  $s$  largest in magnitude elements of  $x$ . It is worth to mention that computing  $g_1(x)$  requires  $O(n)$  (up to a logarithmic factor in  $n$ ) arithmetic operations.

We have an analogous bound for function  $\phi_{\Xi}(x)$ , involved in the definition (5):

**Proposition 2:** Let

$$\Xi_2 = \{\xi \in \mathbb{R}^{n \times n} \mid e^T[\xi]_j = 0, \|\xi\|_1 \leq \varepsilon_j, \|\xi\|_F \leq \varepsilon\},$$

and let  $\|\cdot\| = \|\cdot\|_2$ . Then

$$\phi_{\Xi_2}(x) \leq \varphi_2(x), \quad \varphi_2(x) = \|Px - x\|_2 + \varepsilon g_2(x),$$

where the convex function  $g_2$  satisfies

$$g_2(x) = \min_{x=u+v} \left\{ \|u\|_2 + \sum_j \frac{\varepsilon_j}{\varepsilon} |v|_j \right\}.$$

What was said about the numerical cost of computing  $g_1(x)$  may be repeated here: computing  $g_2(x)$  requires  $O(n)$  (up to “logarithmic factors”) arithmetic operations.

We have a particularly simple and intuitive bound for (6):

**Proposition 3:** Let

$$\Xi_F = \{P + \xi \text{ is stochastic}, \|\xi\|_F \leq \varepsilon\}.$$

Then

$$\phi_{\Xi_F}(x) \leq \varphi_F(x), \quad \varphi_F(x) = \|Px - x\|_2 + \varepsilon \|x\|_2.$$

Note that the upper bound  $\varphi_F(x)$  for  $\phi_{\Xi_F}$  is not very conservative. Indeed, the corresponding worst-case perturbation is

$$\xi^* = \frac{\varepsilon(Px - x)x^T}{\|Px - x\|\|x\|},$$

and for  $F = P + \xi^*$  we have  $\sum_i F_{ij} = 1$ , however the condition  $F_{ij} \geq 0$  (required for  $F$  to be stochastic) may be violated.

The results above can be summarized as follows:

let  $\|\cdot\|_{(1)}$  and  $\|\cdot\|_{(2)}$  be some norms (in the above examples we have  $\|\cdot\|_{(1)}$  set to  $\ell_1$ - or  $\ell_2$ -norm, and  $\|\cdot\|_{(2)} = g_1, g_2$  or  $\|\cdot\|_2$ ). With certain abuse of terminology, we refer to the vector

$$\hat{x} \in \underset{x \in \Sigma}{\text{Argmin}} \left\{ \varphi(x) = \|Px - x\|_{(1)} + \varepsilon \|x\|_{(2)} \right\} \quad (7)$$

as (computable) *robust dominant eigenvector* of the corresponding family of matrices.

Let us discuss some properties of the robust eigenvector.

1.  $\hat{x}$  coincides with  $\bar{x}$  if  $P$  is regular and  $\varepsilon$  is small enough. Recall that stochastic matrix  $P$  is called *regular*, if its largest eigenvalue  $\lambda_1 = 1$  is simple, while all other eigenvalues lay inside the unit circle  $1 - |\lambda_i| \geq \mu > 0$ ,  $i = 2, \dots, n$ . For instance, matrices with all positive entries are regular. Thus dominant eigenvector of a regular matrix is robust with respect to small perturbations. This is the case of matrix  $M$  in (2).
2. In the case  $\|x\|_{(2)} = \|\cdot\|_2$  vector  $\hat{x}$  is unique due to the strict convexity of  $\|x\|_2$  on  $\Sigma$ .
3. For large  $\varepsilon$  vector  $\hat{x}$  is close to  $e \in \Sigma$ ,  $e_i = n^{-1}$ ,  $i = 1, \dots, n$ .

## 4 Solving optimization problem (7)

For irregular matrices or for matrices with small gap  $\mu > 0$  the robust dominant vector  $\hat{x}$  can not be found in explicit form. To compute it one should minimize the function  $\varphi(x)$  over standard simplex  $\Sigma$ . In the above examples these are well structured convex optimization problems, and as such they can be efficiently solved using available optimization software. For instance, medium-size problems (say, when  $n \leq 1.e3 - 1.e4$ ) may be solved using interior-point methods. For large-scale problems one can use methods of mirror-descent family [8, 9, 11] and solve a saddle-point reformulation of (7), as well as various randomized algorithms (see [10, 11] and references therein). Finally, for huge-dimensional problems Yu. Nesterov [12] proposed recently special randomized algorithms which can work with sparse matrices and  $n = 1.e6 - 1.e9$ . We do not discuss these methods here. Instead, we propose a “fast and dirty” method for minimization of (7), which happens to be a close relative of the PageRank technique.

The method starts the minimization process from the point  $x_1$ , which is the minimizer of  $\|x\|_{(2)}$  over  $\Sigma$ . Then we apply the power method with averaging to minimize the first term in  $\varphi(x)$ :

$$x_k = \frac{x_1 + Px_1 + \dots + P^{k-1}x_1}{k}$$

(note that power method without averaging may be non convergent, for instance for a cyclic  $P$ ). We have

$$\|Px_k - x_k\|_{(1)} = \frac{\|P^k x_1 - x_1\|_{(1)}}{k} \leq \frac{c}{k},$$

that is  $\|Px_k - x_k\|_{(1)} = O(1/k)$ .

To fix the ideas, let us consider the case of Proposition 3, where the norms  $\|\cdot\|_{(1)}$  and  $\|\cdot\|_{(2)}$  are the Euclidean norms. We have  $x_1 = e$ , and when writing the method in recurrent form we get

$$x_{k+1} = \left(1 - \frac{1}{k+1}\right) Px_k + \frac{1}{k+1} e. \quad (8)$$

When denoting  $M_k = \left(1 - \frac{1}{k+1}\right) P + \frac{1}{k+1} E$ ,  $E_{ij} = 1/n$ , the iterative method reads  $x_{k+1} = M_k x_k$ ,  $x_1 = e$ . We apply the method until  $\varphi_k = \varphi(x_k)$  starts to arise. Thus we arrive to

*Algorithm 1*

**begin:**  $x_1 = e$

**$k$ -th iteration:**  $x_{k+1} = M_k x_k$ ,

$$M_k = \left(1 - \frac{1}{k+1}\right) P + \frac{1}{k+1} E$$

**stop:**  $\varphi_{k+1} > \varphi_k$ ,  $\bar{x} = x_k$

and  $\bar{x}$  is the approximate solution. Note that the numerical complexity of one iteration is dominated with that of multiplication  $Px$  (and, of course, we should not write  $M_k$  in dense form for sparse  $P$ ).

If we compare Algorithm 1 with PageRank algorithm:

$$x_{k+1} = Mx_k, \quad M = \alpha P + (1 - \alpha)E, \quad \alpha = \text{const},$$

we conclude that our algorithm is PageRank with varying  $\alpha$ , equipped with special stopping rule. The proposed method is also closely related to the regularized algorithm of [13]. However, unlike the method of [13], Algorithm 1 the stopping rule is related to the uncertainty level.

**Choice of  $\varepsilon$**  is an important issue for the described approach. One may consider the following heuristics for Web ranking problems: we may assume that the number  $n_j$  of outgoing links of the  $j$ -th page is known with accuracy  $\pm 1$  for  $qn$  pages, for some  $q < 1$ . Keeping in mind that the average  $n_j = m = 20$  for the whole Web, and taking  $q = 0.5$  we get for uncertainty in  $P$ , measured in Frobenius norm,  $\varepsilon \simeq \sqrt{qn}/m \simeq 0.03\sqrt{n} = 30$  for  $n = 10^6$ . In small problems (like the examples in this paper) the choice  $\varepsilon \simeq 1$  appears to be convenient.

## 5 Numerical results

### 5.1 Simulated graphs

To illustrate an application of the robust eigenvector, proposed above, let us go back to the example in the introduction. We present of Figure 2 the scores computed for this example using three algorithms: 1) PageRank algorithm with  $\alpha = 0.85$ , 2) exact minimization of  $\varphi_F$  with  $\varepsilon = 1$ , 3) Algorithm 1 with  $\varepsilon = 1$ .

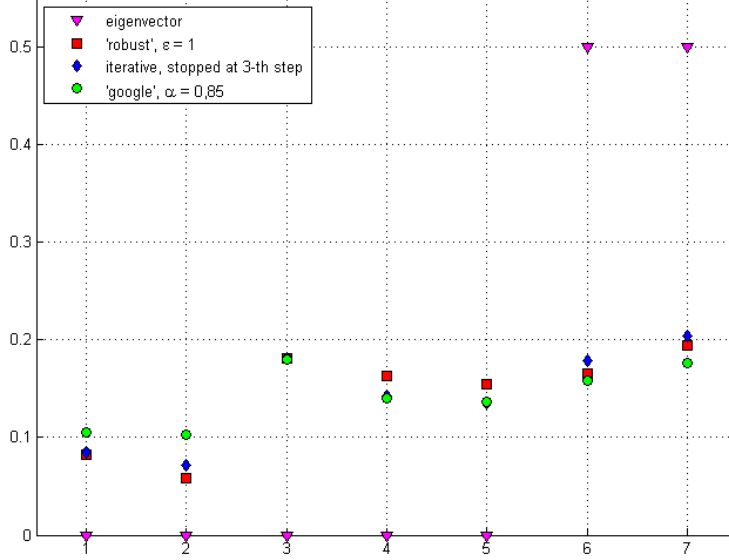


Fig. 2: Results for test example

We observe that the scores for all three methods are close and differ dramatically from the dominant eigenvector of the nominal matrix. The robust approach generates the scores which are acceptable from the “common sense” point of view. It is worth to mentioning that Algorithm 1 requires only 4 iterations ( $\varphi_4 > \varphi_3, \bar{x} = x_3$ ).

We applied the proposed approach to the simulated examples proposed in [13]. The corresponding graphs may be constructed in arbitrary dimensions (from modest to huge ones), with the dominant eigenvectors of matrices  $P$  and  $M$  which can be computed explicitly. Furthermore, the corresponding matrix-vector product  $Px$  can be computed very efficiently without storing the matrix  $P$  itself.

Consider the graph (*Model 1*) of Figure 3. The nodes are shown in circles, the references are denoted by arrows. To get rid of  $(n, n)$  as a dangling node we assume the ability of equally possible jump from this node to any other. There are  $N = n^2$  nodes in the model. We are interested in finding the vector  $x = (x_{ij})$  of scores.

Taking arbitrary value of  $x_{11}$ , say  $x_{11} = 1$ , we get the system of equations equivalent to (1):

$$\begin{cases} x_{i1} = x_{1i} = \frac{1}{2}x_{i-1,1} + 1, & i = 2, \dots, n, & x_{11} = 1, \\ x_{ij} = \frac{1}{2}x_{i-1,j} + \frac{1}{2}x_{i,j-1} + 1, & j, i = 2, \dots, n-1, \\ x_{nj} = x_{jn} = x_{n,j-1} + \frac{1}{2}x_{n-1,j} + 1, & j = 2, \dots, n-1, \\ x_{nn} = x_{n-1,n} + x_{n,n-1} + 1. \end{cases}$$

These equations can be solved explicitly (from  $x_{11}$  we find  $x_{21}$  etc). Then we can proceed to standard normalization and get the scores  $x_{ij}^* = \frac{x_{ij}}{\sum_{i,j} x_{ij}}$ .

Similarly the scores for PageRank matrix  $M$  can be found from equations (it is convenient to take

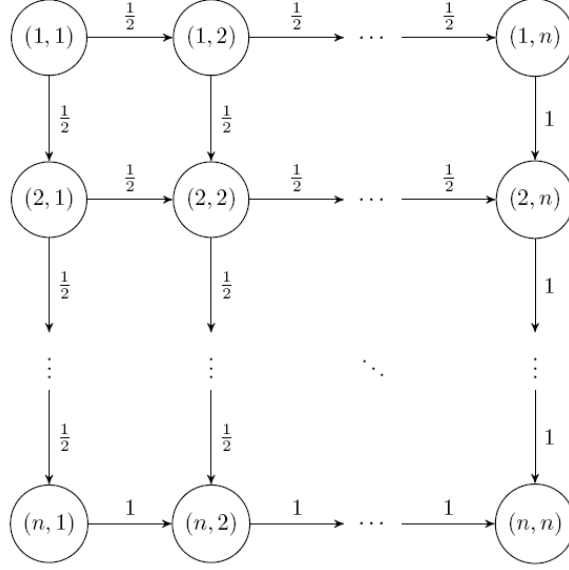


Fig. 3: Model 1

$y_{11} = \alpha$ , then  $y_{nn} = N\alpha$ ):

$$\begin{cases} y_{i1} = y_{1i} = \alpha(\frac{1}{2}y_{i-1,1} + 1), & i = 2, \dots, n, \\ y_{ij} = \alpha(\frac{1}{2}y_{i-1,j} + \frac{1}{2}y_{i,j-1} + 1), & j, i = 2, \dots, n-1, \\ y_{nj} = y_{jn} = \alpha(y_{n,j-1} + \frac{1}{2}y_{n-1,j} + 1), & j = 2, \dots, n-1, \\ y_{nn} = \alpha(y_{n-1,n} + y_{n,n-1} + 1) \end{cases}$$

and normalization  $y_{ij}^* = \frac{y_{ij}}{\sum_{i,j} y_{ij}}$ .

Note that even very small perturbations may strongly corrupt the graph on Figure 3. Thus, taking large penalty parameter  $\varepsilon$ , as it was suggested in the discussion at the end of Section 4 would lead to complete “equalization” of the scores. That is why we choose small penalty coefficient in this example (specifically, it was taken  $\varepsilon = 0.01$ ).

Let us denote  $x_\alpha = y^*$ . We compare now the score vector  $x^*$  for nominal  $P$  with the score  $x_\alpha$  by PageRank and the results obtained by Algorithm 1. On Figure 4 we present the scores of diagonal nodes, i.e.  $x_{i,i}, i = 1, \dots, n$  for  $n = 200$  (that is  $N = 40000$ ) and  $\alpha = 0.85$ . Figure 5 shows the scores  $x_{ni}, i = 1, \dots, n$  of the last row. From the geometry of our graph (Figure 3) it is natural to expect the “true” scores  $x_{ij}$  to increase with  $i, j$  and the highest scores should be achieved in south-east corner of the square. The score vector, obtained as dominant eigenvector of  $P$ , possesses these properties. PageRank solution and robust solution provide both strongly “equalized” ranks; the robust solution appears to better fit the expected one.

Let us modify slightly the graph of Model 1: in the new *Model 2* we link the node  $(n, n)$  *only* to the node  $(1, 1)$ . For this model the system of equations for which defines the nominal scores is similar to the system for Model 1, and the dominant eigenvector of  $P$  and PageRank vector  $x^\alpha$  can be computed explicitly. On the other hand, it is obvious that the power method would not converge for this model. Indeed if  $x_{11}^0 = 1, x_{ij}^0 = 0, \{ij\} \neq \{11\}$  then after  $2n - 1$  iterations we return at the same state, and matrix  $P$  is cyclic.

The results for Model 2 with the same parameters ( $n = 200, \varepsilon = 0.01$ ) are presented at Figures 6 and 7. In this example, same as for Model 1, the score vector obtained using PageRank is close to the robust one. And may be robust scores are slightly closer to our expectations of “common sense” ranks.

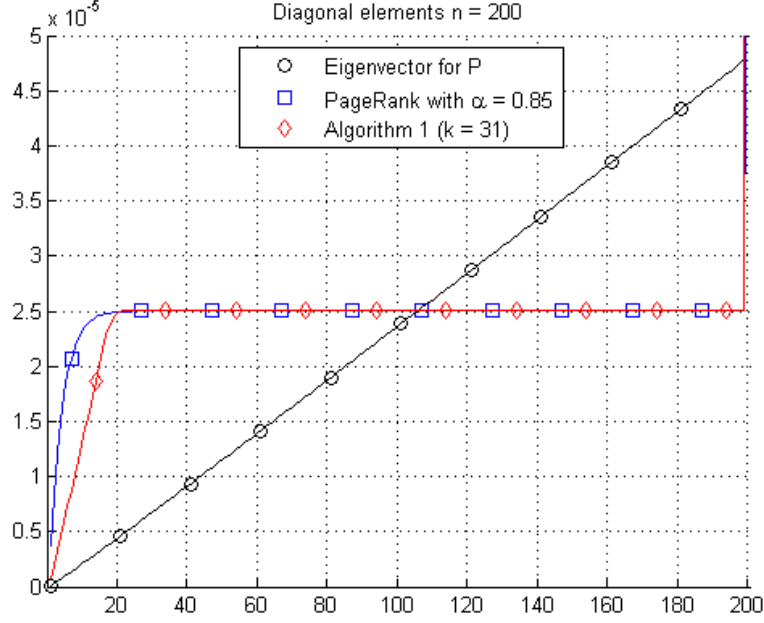


Fig. 4: Diagonal scores

## 5.2 “Real-life” data

We have tested the proposed techniques on the experimental datasets, used in experiments on link analysis ranking algorithms described in [2]. We present here experimental results for adjacency matrices from two refined datasets: **Computational Complexity** (CompCom) and **Movies**, downloaded from <http://www.cs.toronto.edu/~tsap/experiments/>. **CompCom** data represent 789 pages with totalize 1449 links (with average number of links per page is equal to 1.8365). **Movies** dataset has 4754 notes and 17198 (average number of links per page is 3.6176). We compare high accuracy solution to the problem (7) (with the norms  $\|\cdot\|_{(1)}$  and  $\|\cdot\|_{(2)}$  are Euclidean norms) obtained using *MOSEK* optimization software [1] with the scores by Algorithm 1 and those obtained by the PageRank algorithm. The same value of penalization parameter  $\varepsilon = 1$  in (7) is used in all experiments. On Figure 8 we present the 20 largest elements of the score vectors for the **CompCom** data. In this experiment the optimal value of the problem (7) is 0.0587. The corresponding value for solution supplied by Algorithm 1 (after 3 iterations) is 0.0756. It should be noted that the high scores  $x_{645}^\alpha$  and  $x_{654}^\alpha$  by PageRank are pure artifacts – they correspond to the pages which receive numerous links but only reference each other. The techniques proposed in this paper seem better handle this sort of problem than the classical PageRank algorithm.

Figure 9 shows the 20 largest elements of the score vectors for the **Movies** data. In this case the best scores computed by 3 competitors are quite close. The optimal value of the problem (7) is 0.0288; the approximate solution supplied by Algorithm 1 (after 4 iterations) attains the value 0.0379.

## 6 Directions for future research

1. *Structured perturbations.* It is often natural to assume that the perturbation  $\xi$  of nominal matrix  $P$  possesses some structure. For instance, we can consider  $\xi = M\Delta N$ , where  $\|\Delta\|_F \leq \varepsilon$ , and  $M, N$  are rectangular matrices of appropriate dimensions, or  $\xi = \sum t_i A_i$ , where  $\|t\|_2 \leq \varepsilon$ , with fixed matrices  $A_i \in \mathbb{R}^{n \times n}$ .
2. *Implicit uncertainty model.* Family  $\mathcal{P}$  can be defined implicitly. Suppose that we can sample randomly from  $\mathcal{P}$ , so that *random* realizations  $F_i \in \mathcal{P}$ ,  $i = 1, \dots, N$  are available. One may try to find  $x$  which is approximate dominant eigenvector for all of them. For instance, one can fix  $\varepsilon > 0$



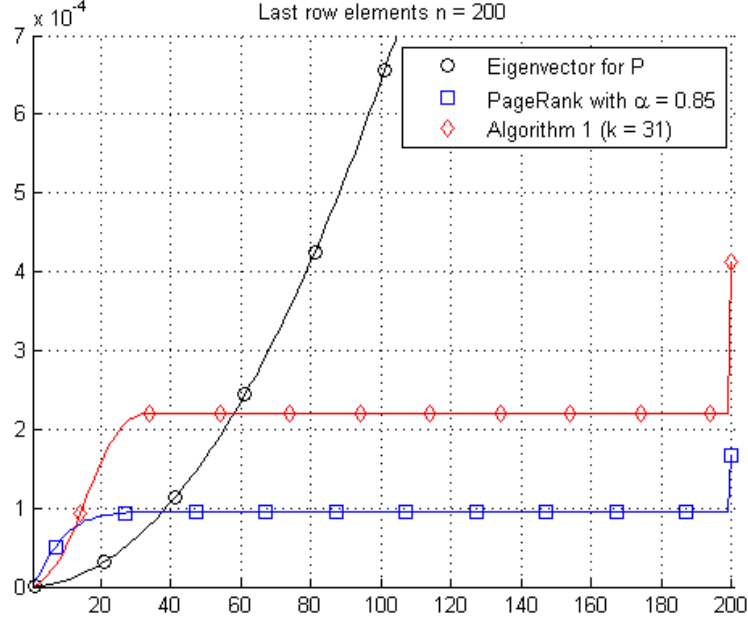


Fig. 5: Last row scores

and look for solution of a system of convex inequalities

$$\|F_i x - x\| \leq \varepsilon, \quad i = 1, \dots, N, \quad x \in \Sigma.$$

Such system can be solved iteratively using the mirror-descent algorithm. This approach is close to [14, 15].

3. *Computational research.* The development of numerical methods tailored for optimization problem (7) and their comparison is of great interest.
4. *Real-life examples.* We plan to test extensively the proposed approach on the large-scale Web rank problems data available from different Internet databases.

## 7 ACKNOWLEDGMENTS

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### .1 Proof of Propositions 1 and 2

Note that

$$\begin{aligned} \phi_{\Xi_1}(x) &= \max_{\xi \in \Xi_1} \|(\xi + P - I)x\|_1 \\ &= \max_{\xi \in \Xi_1} \max_{u \in \mathbb{R}^n, \|u\|_\infty \leq 1} u^T (\xi + P - I)x. \end{aligned}$$

Then  $z = \xi^T u$  satisfies

$$\begin{aligned} |z_j| &= \left| \sum_i \xi_{ij} u_i \right| \leq \sum_i |\xi_{ij}| \leq \varepsilon_j. \\ \|z\|_1 &= \sum_j \left| \sum_i \xi_{ij} u_i \right| \leq \sum_j \sum_i |\xi_{ij}| \leq \varepsilon. \end{aligned}$$

We conclude that

$$\begin{aligned}\phi_{\Xi_1}(x) &\leq \varphi_1(x) := \max_{u \in \mathbb{R}^n, \|u\|_\infty \leq 1} u^T (P - I)x + \varepsilon g_1(x) \\ &= \|(P - I)x\|_1 + \varepsilon g_1(x),\end{aligned}\tag{9}$$

where

$$\begin{aligned}g_1(x) &= \varepsilon^{-1} \max_{z \in \mathbb{R}^n, \|z\|_\infty \leq \varepsilon, |z|_j \leq \varepsilon_j} z^T x \\ &= \max_{z \in \mathbb{R}^n, \|z\|_\infty \leq 1, |z|_j \leq \varepsilon_j / \varepsilon} z^T x.\end{aligned}\tag{10}$$

Note that (10) is a strictly feasible conic optimization problem, when dualizing the constraints we come to

$$g_1(x) = \min_{u+v=x} \{ \|u\|_\infty + \sum_j \frac{\varepsilon_j}{\varepsilon} |v_j| \}.$$

Along with (9) this implies Proposition 1.  $\square$

The proof of Proposition 2 is completely analogous.

## .2 Proof of Proposition 3

We start with the following lemma, which is a modification of Theorem 3.1 in [6].

**Lemma 1:** For  $a \in \mathbb{R}^n, b \in \mathbb{R}^m, \xi \in \mathbb{R}^{n \times m}$  it holds

$$\max_{\|\xi\|_F \leq \varepsilon} \|a + \xi b\|_2 = \|a\|_2 + \varepsilon \|b\|_2.$$

*Proof:* Indeed,  $\|a + \xi b\|_2 \leq \|a\|_2 + \varepsilon \|b\|_2$ , while for  $a \neq 0$ ,  $\xi = \xi^* = \frac{\varepsilon a b^T}{\|a\|_2 \|b\|_2}$  equality holds and  $\|\xi^*\|_F = \varepsilon$  (for  $a = 0$  one can take  $\xi^* = \frac{\varepsilon c b^T}{\|b\|_2}$ ,  $\|c\|_2 = 1, c \in \mathbb{R}^n$  arbitrary).  $\square$

We can now bound

$$\begin{aligned}\max_{\xi \in \Xi_F} \|(P + \xi)x - x\|_2 &= \max_{P + \xi \in \mathcal{S}, \|\xi\|_F \leq \varepsilon} \|(P + \xi)x - x\| \\ &\leq \max_{\|\xi\|_F \leq \varepsilon} \|Px - x + \xi x\|_2 = \|Px - x\|_2 + \varepsilon \|x\|_2,\end{aligned}$$

where the last equality is due to Lemma 1.  $\square$

## References

- [1] E. Andersen, D. Andersen *MOSEK Optimization Tools Manual* <http://docs.mosek.com/6.0/tools/index.html>.
- [2] A. Borodin, G. O. Roberts, J. S. Rosenthal, P. Tsaparas, Finding Authorities and Hubs from Link Structures on the World Wide Web, In *10th International World Wide Web Conference*, May 2000.
- [3] M. Franceschet, PageRank: standing on the shoulders of giants, *Communications of the ACM*, 54 (6): 92–101, 2011.
- [4] S. Brin and L. Page, The anatomy of a large-scale hypertextual web search engine, *Comput. Netw. ISDN Syst.*, 30(1-7):107–117, 1998.
- [5] A.N. Langville and C.D. Meyer, *Google's PageRank and Beyond: The Science of Search Engine Rankings*, Princeton University Press, 2006.
- [6] L. El Ghaoui, H. Lebret, Robust solutions to least-squares problems with uncertain data, *SIAM J. Matrix Anal. Appl.*, 18(4):1035–1064, 1997.
- [7] A. Ben-Tal, A. Nemirovski, Robust convex optimization, *Math. Oper. Res.*, 23(4):769–805, 1998.

- [8] A.S.Nemirovskij, D. B. Yudin, *Problem Complexity and Method Efficiency in Optimization*, John Wiley, New York, 1983.
- [9] G. Lan, A.S. Nemirovskij, A. Shapiro, Validation analysis of mirror descent stochastic approximation method, *Math. Progr.*, DOI 10.1007/s10107-011-0442-6, 2011.
- [10] H. Ishii and R. Tempo, Distributed Randomized Algorithms for the PageRank Computation, *IEEE Trans. on Automatic Control*, 55(9):1987–2002, 2010.
- [11] A.V. Nazin, B.T. Polyak, Adaptive Randomized Algorithm for Finding Eigenvector of Stochastic Matrix with Application to PageRank, *48th IEEE Conference on Decision and Control*, Shanghai, China, December 2009.
- [12] Yu. Nesterov, Subgradient methods for huge-scale optimization problems, *CORE Discussion Paper*, No 02, 2012.
- [13] B.T. Polyak, A.V. Timonina, PageRank: new regularizations and simulation models, *18th IFAC World Congress*, 11202–11207, Milan, Italy, August 2011.
- [14] B.T. Polyak, Random Algorithms for Solving Convex Inequalities, *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, (eds. D.Butnariu, Y.Censor and S.Reich), Elsevier, 409–422, 2001.
- [15] B.T. Polyak, R. Tempo, Probabilistic Robust Design with Linear Quadratic Regulators, *Systems and Control Letters*, **43**, No. 5, 343–353, 2001.

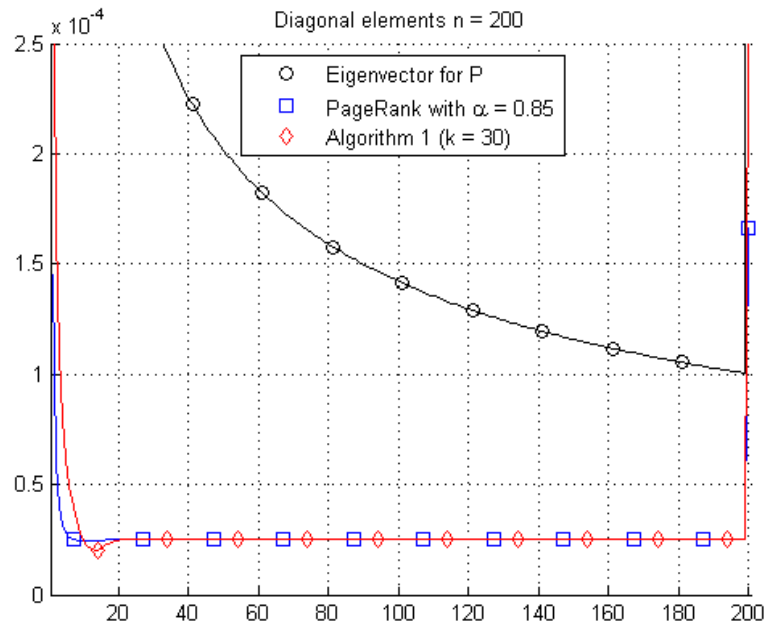


Fig. 6: Diagonal scores for the second model.

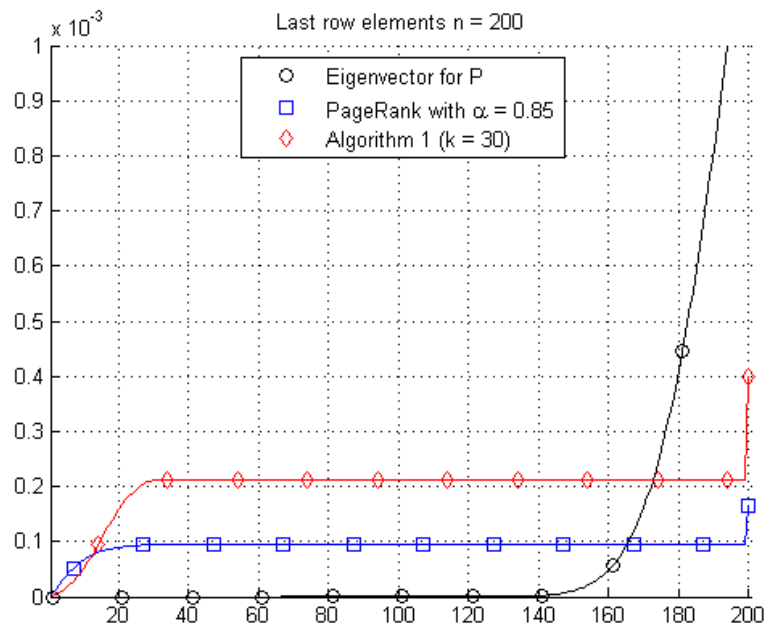


Fig. 7: Last row scores for the second model.

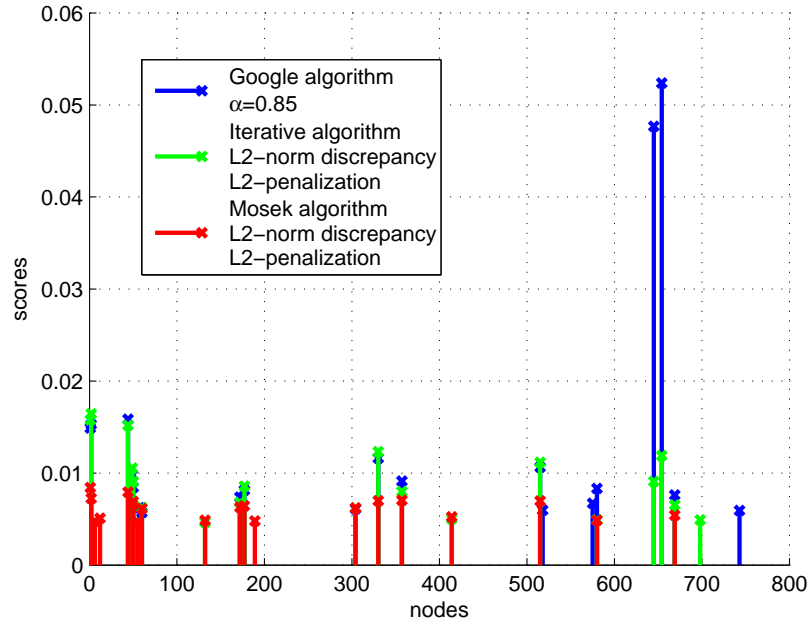


Fig. 8: 20 highest scores for CompCom matrix

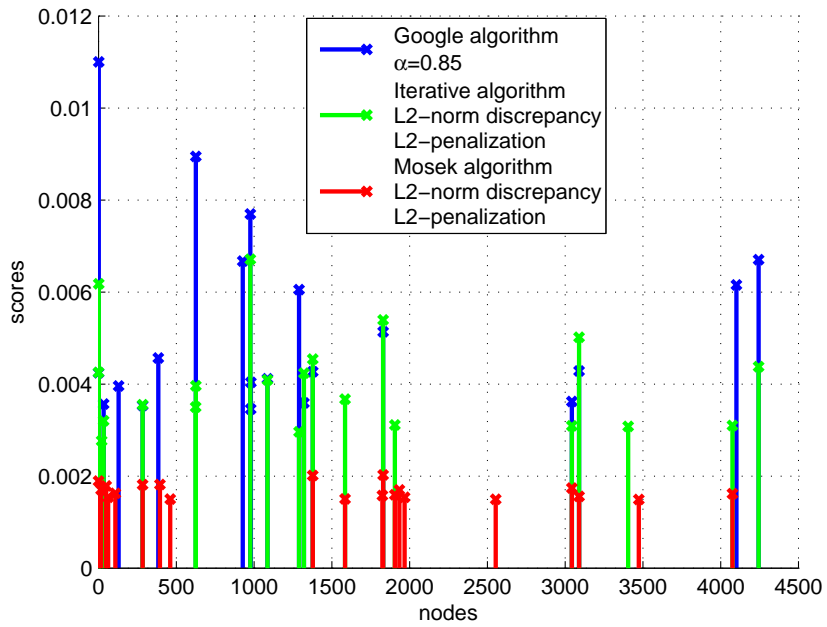


Fig. 9: 20 highest scores for Movies matrix